Goéland: A Concurrent Tableau-Based ATP that Produces Machine-Checkable Proofs

EuroProofNet School on Natural Formal Mathematics

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Computer-Assisted Proofs

Automated Theorem Proving

- Click-and-prove software
- Autonomous search
- Statement or proof-like trace

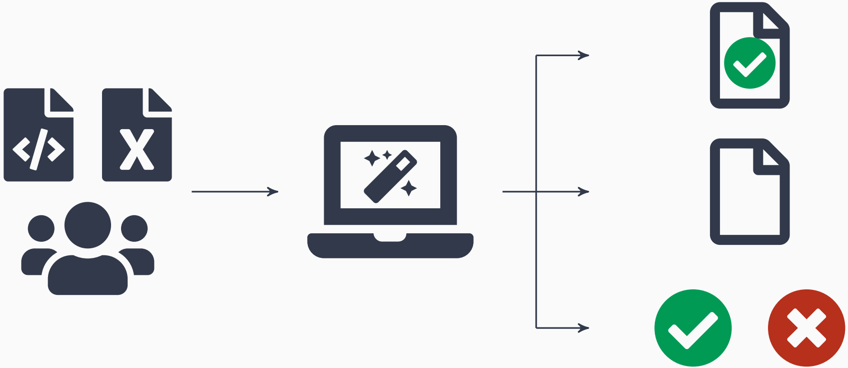
Interactive Theorem Proving

- Proof assistants
- Guided search
- Machine-checkable proofs

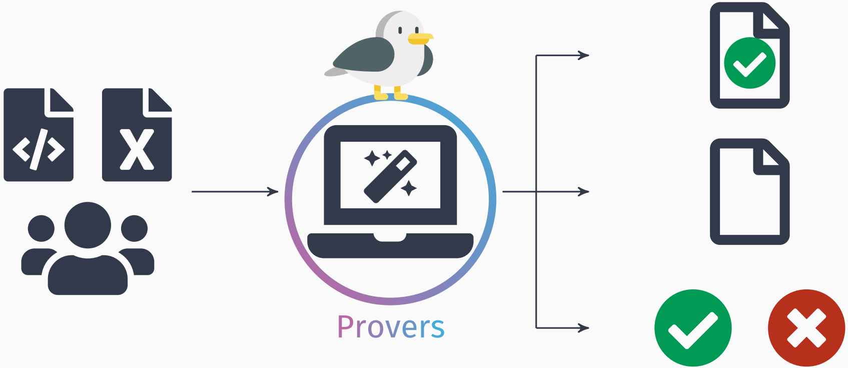
Yes/No answer	Proof	Certificate

Trust Scale

Big Picture



Big Picture



1. Preliminary Notions

Context

First-Order Logic (FOL)

- Expressivity: elements and properties about them
- Semi-decidable
- Efficient reasoning methods

Tableaux: Origin & Strengths

- Beth and Hintikka
- Extended by Smullyan and Fitting
- Unaltered original formula
- Output a proof

Method of Analytic Tableaux

Principle

- A set of axioms and one conjecture
- Refutation
- Syntactic rules: $\odot, \alpha, \delta, \beta, \gamma$
- Close all the branches

$$
\dfrac{\neg(\exists x.\ P(x) \Rightarrow (P(a) \land P(b)))}{\dfrac{\neg(P(a) \Rightarrow (P(a) \land P(b)))}{\dfrac{P(a), \neg(P(a) \land P(b))}{\dfrac{\neg P(a)}{\odot}\,\odot \qquad \dfrac{\neg P(b)}{\dfrac{\neg(P(b) \Rightarrow (P(a) \land P(b)))}{\dfrac{P(b), \neg(P(a) \land P(b))}{\odot}\,\odot}\,\alpha_{\neg\Rightarrow}}\,\gamma_{\neg\exists}}\,\beta_{\neg\land}}\,\alpha_{\neg\Rightarrow}}\,\gamma_{\neg\exists}
$$

Tableau-Based Proof-Search Procedure

Rules

- $\odot$: Closure rule
- α, β: Expand the tree
- γ: Free variables
- δ: Skolemization

Tableaux in AR

- Free variables
- Substitutions (local & global)

$$\frac{Human(Socrates), \neg Human(Socrates)}{\odot}\ \odot$$

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$$Human(Socrates)$$
$$\forall x.\ \neg Human(x)$$

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$$\frac{\begin{array}{c} Human(Socrates) \\ \forall x.\ \neg Human(x) \end{array}}{\neg Human(X)}\ \gamma_\forall$$

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- Free variables
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$$\dfrac{\dfrac{\begin{array}{c} Human(Socrates) \\ \forall x.\ \neg Human(x) \end{array}}{\neg Human(Socrates)}\ \gamma\forall}{\sigma = \{X \mapsto Socrates\}}\ \odot_\sigma$$

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- $\odot$: Closure rule
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- γ: Free variables
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Tableaux in AR

- Free variables
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$$\forall x.\ P(x)$$
$$\neg P(a) \vee \neg P(b)$$

Tableau-Based Proof-Search Procedure

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- δ: Skolemization

Tableaux in AR

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$$\forall x.\ P(x)$$

$$\frac{\neg P(a) \vee \neg P(b)}{P(X)}\ \gamma\forall$$

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Rules

- $\odot$: Closure rule
- α, β: Expand the tree
- γ: Free variables
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Tableaux in AR

- Free variables
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$$\dfrac{\dfrac{\begin{array}{c}\forall x.\ P(x) \\ \neg P(a) \vee \neg P(b)\end{array}}{P(X)}\ \gamma_\forall}{\neg P(a) \qquad\qquad \neg P(b)}\ \beta_\vee$$

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$$
\cfrac{\cfrac{\begin{array}{c}\forall x.\ P(x) \\ \neg P(a) \vee \neg P(b)\end{array}}{P(X)}\ \gamma_\forall}{\neg P(a) \qquad\qquad\qquad \neg P(b)}\ \beta_\vee
$$

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- Free variables
- Substitutions (local & global)

$$\dfrac{\dfrac{\begin{array}{c}\forall x.\ P(x) \\ \neg P(a) \lor \neg P(b)\end{array}}{P(X)}\,\gamma_\forall}{\neg P(a) \qquad\qquad \neg P(b)}\,\beta_\lor$$

Tableau-Based Proof-Search Procedure

Rules

- $\odot$: Closure rule
- α, β: Expand the tree
- γ: Free variables
- δ: Skolemization

Tableaux in AR

- Free variables
- Substitutions (local & global)

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\forall x.\ P(x) \\ \neg P(a) \vee \neg P(b)\end{array}}{P(X)}\ \gamma_\forall}{\cfrac{\neg P(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad \neg P(b)}\ \beta_\vee}{}$$

Tableau-Based Proof-Search Procedure

Rules

- $\odot$: Closure rule
- α, β: Expand the tree
- γ: Free variables
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Tableaux in AR

- Free variables
- Substitutions (local & global)

$$\dfrac{\dfrac{\begin{array}{c}\forall x.\ P(x)\\ \neg P(a) \vee \neg P(b)\end{array}}{P(a)}\,\gamma_{\forall}}{\dfrac{\neg P(a)}{\sigma = \{X \mapsto a\}}\,\odot_{\sigma} \qquad\qquad \neg P(b)}\,\beta_{\vee}$$

Tableau-Based Proof-Search Procedure

Rules

- $\odot$: Closure rule
- α, β: Expand the tree
- γ: Free variables
- δ: Skolemization

Tableaux in AR

- Free variables
- Substitutions (local & global)

$$\forall x.\ P(x)$$

$$\cfrac{\cfrac{\neg P(a) \lor \neg P(b)}{P(a)}\ \gamma_\forall}{\cfrac{\neg P(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad\qquad \cfrac{\neg P(b)}{P(X_2)}\ \gamma_\forall}\ \beta_\lor$$

Tableau-Based Proof-Search Procedure

Rules

- $\odot$: Closure rule
- α, β: Expand the tree
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- δ: Skolemization

Tableaux in AR

- Free variables
- Substitutions (local & global)

$$\forall x.\ P(x)$$

$$\dfrac{\neg P(a) \lor \neg P(b)}{P(a)}\ \gamma_\forall$$

$$\dfrac{}{\dfrac{\neg P(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad\qquad \dfrac{\neg P(b)}{P(X_2)}\ \gamma_\forall}\ \beta_\lor$$

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$$\dfrac{\dfrac{\begin{array}{c}\forall x.\ P(x)\\ \neg P(a) \lor \neg P(b)\end{array}}{P(a)}\,\gamma_\forall}{\dfrac{\neg P(a)}{\sigma = \{X \mapsto a\}}\,\odot_\sigma \qquad \dfrac{\dfrac{\neg P(b)}{P(X_2)}\,\gamma_\forall}{\sigma = \{X_2 \mapsto b\}}\,\odot_\sigma}\,\beta_\lor$$

Tableau-Based Proof-Search Procedure

Rules

- $\odot$: Closure rule
- α, β: Expand the tree
- γ: Free variables
- δ: Skolemization

Tableaux in AR

- Free variables
- Substitutions (local & global)

$$\dfrac{\dfrac{\begin{array}{c}\forall x.\ P(x)\\ \neg P(a) \lor \neg P(b)\end{array}}{P(a)}\;\gamma_\forall}{\dfrac{\neg P(a)}{\sigma = \{X \mapsto a\}}\;\odot_\sigma \qquad \dfrac{\dfrac{\neg P(b)}{P(b)}\;\gamma_\forall}{\sigma = \{X_2 \mapsto b\}}\;\odot_\sigma}\;\beta_\lor$$

2. Fairness Management in Tableau Proof-Search Procedure: a Concurrent Approach

Fairness Management

Unfairness Sources

- The selection of a branch B
- Determining whether B should be closed or expanded
- If B is to be closed, the choice of a pair of complementary literals and thus a closing substitution
- If B is to be expanded, the selection of a formula to which an expansion rule is applied

Fairness Management

Unfairness Sources

- The selection of a branch B
- Determining whether B should be closed or expanded
- If B is to be closed, the choice of a pair of complementary literals and thus a closing substitution
- If B is to be expanded, the selection of a formula to which an expansion rule is applied

State-of-the-Art Answers & Heuristics

- Limit the number of application of γ-rules
- Iterative deepening
- Rules ordering ($\odot \prec \alpha \prec \delta \prec \beta \prec \gamma$)

Motivating Example

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\forall x.\ ((P(x) \vee Q(x)) \wedge \forall y.\ S(x))$$

Motivating Example

$$\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\, ((P(x) \vee Q(x)) \wedge \forall y.\, S(x))\end{array}}{\begin{array}{c}P(X) \vee Q(X)\\ \forall y.\, S(X)\end{array}}\ \gamma_\forall + \alpha_\wedge$$

Motivating Example

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(X) \lor Q(X)} \ \gamma_\forall + \alpha_\land$$
$$\dfrac{\forall y.\ S(X)}{\begin{array}{ccc} P(X) & \qquad\qquad & Q(X) \\ \forall y.\ S(X) & & \forall y.\ S(X) \end{array}} \ \beta_\lor$$

Motivating Example

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \vee Q(x)) \wedge \forall y.\ S(x))}{\begin{array}{c} P(X) \vee Q(X) \\ \forall y.\ S(X) \end{array}}\ \gamma_\forall + \alpha_\wedge$$

$$\dfrac{}{\begin{array}{ccc} P(X) & \qquad\qquad & Q(X) \\ \forall y.\ S(X) & & \forall y.\ S(X) \end{array}}\ \beta_\vee$$

Motivating Example

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\begin{array}{c} P(X) \lor Q(X) \\ \forall y.\ S(X) \end{array}}\ \gamma_\forall + \alpha_\land$$

$$\dfrac{}{\begin{array}{c} P(X) \\ \forall y.\ S(X) \end{array} \qquad\qquad \begin{array}{c} Q(X) \\ \forall y.\ S(X) \end{array}}\ \beta_\lor$$

$$\dfrac{\begin{array}{c} P(X) \\ \forall y.\ S(X) \end{array}}{\sigma = \{X \mapsto a\}}\ \odot_\sigma$$

Motivating Example

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(a) \lor Q(a)}\ \gamma_{\forall} + \alpha_{\land}$$

$$\forall y.\ S(a)$$

$$\dfrac{\qquad\qquad\qquad\qquad}{\begin{array}{ccc} \dfrac{\begin{array}{c} P(a) \\ \forall y.\ S(a) \end{array}}{\sigma = \{X \mapsto a\}}\ \odot_{\sigma} & \qquad\qquad & \begin{array}{c} Q(a) \\ \forall y.\ S(a) \end{array} \end{array}}\ \beta_{\lor}$$

Motivating Example

$$
\begin{array}{c}
\neg P(a) \\
\neg Q(b) \\
\neg S(c) \\
\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(a) \lor Q(a)}\ \gamma_\forall + \alpha_\land \\
\dfrac{\forall y.\ S(a)}{\begin{array}{cc} P(a) & Q(a) \\ \dfrac{\forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma & \dfrac{\boxed{\forall y.\ S(a)}}{S(a)}\ \gamma_\forall \end{array}}\ \beta_\lor
\end{array}
$$

Motivating Example

$$
\begin{array}{c}
\neg P(a)\\
\neg Q(b)\\
\neg S(c)\\
\dfrac{\boxed{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}}{\dfrac{\begin{array}{c} P(a) \lor Q(a) \\ \forall y.\ S(a) \end{array}}{\begin{array}{cc} \dfrac{\begin{array}{c} P(a) \\ \forall y.\ S(a) \end{array}}{\sigma = \{X \mapsto a\}}\ \odot_\sigma & \dfrac{\dfrac{\begin{array}{c} Q(a) \\ \forall y.\ S(a) \end{array}}{S(a)}\ \gamma_\forall}{\begin{array}{c} P(X_2) \lor Q(X_2) \\ \forall y.\ S(X_2) \end{array}}\ \gamma_\forall + \alpha_\land \end{array}}\ \beta_\lor}\ \gamma_\forall + \alpha_\land
\end{array}
$$

Motivating Example

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(a) \lor Q(a)}\ \gamma_\forall + \alpha_\land$$
$$\dfrac{\forall y.\ S(a)}{P(a) \qquad\qquad\qquad Q(a)}\ \beta_\lor$$

$$\dfrac{\begin{array}{c} P(a) \\ \forall y.\ S(a) \end{array}}{\sigma = \{X \mapsto a\}}\ \odot_\sigma$$

$$Q(a)$$
$$\dfrac{\forall y.\ S(a)}{S(a)}\ \gamma_\forall$$
$$\dfrac{}{P(X_2) \lor Q(X_2)}\ \gamma_\forall + \alpha_\land$$
$$\dfrac{\forall y.\ S(X_2)}{\begin{array}{cc} P(X_2) & Q(X_2) \\ \forall y.\ S(X_2) & \forall y.\ S(X_2) \end{array}}\ \beta_\lor$$

Motivating Example

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\begin{array}{c} P(a) \lor Q(a) \\ \forall y.\ S(a) \end{array}}\ \gamma_\forall + \alpha_\land$$

$$\dfrac{\quad}{\begin{array}{ccc} P(a) & \qquad & Q(a) \\ \forall y.\ S(a) & & \forall y.\ S(a) \end{array}}\ \beta_\lor$$

$$\dfrac{\forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma$$

$$\dfrac{\forall y.\ S(a)}{S(a)}\ \gamma_\forall$$

$$\dfrac{S(a)}{\begin{array}{c} P(X_2) \lor Q(X_2) \\ \forall y.\ S(X_2) \end{array}}\ \gamma_\forall + \alpha_\land$$

$$\dfrac{\quad}{\begin{array}{ccc} P(X_2) & \qquad & Q(X_2) \\ \forall y.\ S(X_2) & & \forall y.\ S(X_2) \end{array}}\ \beta_\lor$$

$$\dfrac{\forall y.\ S(X_2)}{\cdots} \qquad \dfrac{\forall y.\ S(X_2)}{\cdots}$$

Motivating Example — A Better Heuristic

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\forall x.\ ((P(x) \vee Q(x)) \wedge \forall y.\ S(x))$$

Motivating Example — A Better Heuristic

$$\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ ((P(x) \vee Q(x)) \wedge \forall y.\ S(x))\end{array}}{\begin{array}{c}P(X) \vee Q(X)\\ \forall y.\ S(X)\end{array}}\ \gamma_\forall + \alpha_\wedge$$

Motivating Example — A Better Heuristic

$$
\cfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))\end{array}}{\cfrac{\begin{array}{c}\boxed{P(X) \lor Q(X)}\\ \forall y.\ S(X)\end{array}}{\begin{array}{c}P(X)\\ \forall y.\ S(X)\end{array} \qquad \begin{array}{c}Q(X)\\ \forall y.\ S(X)\end{array}}\ \beta_\lor}\ \gamma_\forall + \alpha_\land
$$

Motivating Example — A Better Heuristic

$$
\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))\end{array}}{\dfrac{\begin{array}{c}P(X) \lor Q(X)\\ \forall y.\ S(X)\end{array}}{\begin{array}{c}\dfrac{\begin{array}{c}P(X)\\ \forall y.\ S(X)\end{array}}{S(X)}\ \gamma_\forall \end{array} \qquad \begin{array}{c}Q(X)\\ \forall y.\ S(X)\end{array}}\ \beta_\lor}\ \gamma_\forall + \alpha_\land
$$

Motivating Example — A Better Heuristic

$$\neg P(a)$$
$$\neg Q(b)$$
$$\boxed{\neg S(c)}$$
$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(X) \lor Q(X)}\ \gamma_\forall + \alpha_\land$$
$$\dfrac{\forall y.\ S(X)}{P(X) \qquad\qquad Q(X)}\ \beta_\lor$$
$$\dfrac{\forall y.\ S(X)}{\boxed{S(X)}}\ \gamma_\forall \qquad\qquad \forall y.\ S(X)$$
$$\dfrac{}{\sigma = \{X \mapsto c\}}\ \odot_\sigma$$

Motivating Example — A Better Heuristic

$$
\begin{array}{c}
\neg P(a) \\
\neg Q(b) \\
\neg S(c) \\
\dfrac{\forall x.\, ((P(x) \lor Q(x)) \land \forall y.\, S(x))}{P(c) \lor Q(c)}\ \gamma_\forall + \alpha_\land \\
\forall y.\, S(c) \\
\hline
\begin{array}{cc}
P(c) & Q(c) \\
\dfrac{\dfrac{\forall y.\, S(c)}{S(c)}\ \gamma_\forall}{\sigma = \{X \mapsto c\}}\ \odot_\sigma & \forall y.\, S(c)
\end{array}
\end{array}
\ \beta_\lor
$$

$$
\begin{array}{c}
\neg P(a)\\
\neg Q(b)\\
\neg S(c)\\
\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(c) \lor Q(c)}\ \gamma_\forall + \alpha_\land\\
\dfrac{\forall y.\ S(c)}{P(c) \qquad\qquad\qquad Q(c)}\ \beta_\lor
\end{array}
$$

$$
\dfrac{\dfrac{\forall y.\ S(c)}{S(c)}\ \gamma_\forall}{\sigma = \{X \mapsto c\}}\ \odot_\sigma
\qquad\qquad
\dfrac{\forall y.\ S(c)}{S(c)}\ \gamma_\forall
$$

Motivating Example — A Better Heuristic

$$
\begin{array}{c}
\neg P(a)\\
\neg Q(b)\\
\neg S(c)\\
\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(c) \lor Q(c)}\ \gamma_\forall + \alpha_\land\\
\dfrac{\forall y.\ S(c)}{\begin{array}{cc}
P(c) & Q(c)\\
\dfrac{\dfrac{\forall y.\ S(c)}{S(c)}\ \gamma_\forall}{\sigma = \{X \mapsto c\}}\ \odot_\sigma &
\begin{array}{c}\dfrac{\forall y.\ S(c)}{S(c)}\ \gamma_\forall\\ \dfrac{}{\odot}\ \odot_\sigma\end{array}
\end{array}}\ \beta_\lor
\end{array}
$$

Exploring Branches in Parallel?

Approach

- Each branch searches for a local solution
- Management of multiple solutions with successive attempts and backtracking
- Forbid previously tried solutions
- Iterative deepening, limit of γ-rule and rules ordering

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New Challenges

- Free variable dependency
- Communication between branches

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\begin{array}{c} P(X) \lor Q(X) \\ \forall y.\ S(X) \end{array}}\ \gamma_\forall + \alpha_\land$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\forall x.\ ((P(x) \vee Q(x)) \wedge \forall y.\ S(x))}{\boxed{P(X) \vee Q(X)}} \ \gamma_\forall + \alpha_\wedge$$
$$\dfrac{\forall y.\ S(X)}{\begin{array}{cc} P(X) & Q(X) \\ \forall y.\ S(X) & \forall y.\ S(X) \end{array}} \ \beta_\vee$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(X) \lor Q(X)}\ \gamma_\forall + \alpha_\land$$

$$\dfrac{\forall y.\ S(X)}{P(X) \qquad\qquad Q(X)}\ \beta_\lor$$

$$\dfrac{\forall y.\ S(X)}{\odot}\ \odot_\sigma \qquad\qquad \dfrac{\forall y.\ S(X)}{\odot}\ \odot_\sigma$$

$$\sigma = \{X \mapsto a\} \qquad\qquad \sigma = \{X \mapsto b\}$$

Solving Fairness Issues with Concurrency

$$\begin{array}{c} \neg P(a) \\ \neg Q(b) \\ \neg S(c) \\ \forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x)) \end{array}$$

$$\frac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\begin{array}{c} P(X) \lor Q(X) \\ \forall y.\ S(X) \end{array}}\ \gamma_\forall + \alpha_\land$$

$$\frac{\begin{array}{c} P(X) \lor Q(X) \\ \forall y.\ S(X) \end{array}}{\begin{array}{c} P(X) \\ \forall y.\ S(X) \end{array} \qquad \begin{array}{c} Q(X) \\ \forall y.\ S(X) \end{array}}\ \beta_\lor$$

$$\frac{\begin{array}{c} P(X) \\ \forall y.\ S(X) \end{array}}{\odot}\ \odot_\sigma \qquad \frac{\begin{array}{c} Q(X) \\ \forall y.\ S(X) \end{array}}{\odot}\ \odot_\sigma$$

$$\sigma = \{X \mapsto a\} \qquad \sigma = \{X \mapsto b\}$$

Solving Fairness Issues with Concurrency

$$
\begin{array}{c}
\neg P(a)\\
\neg Q(b)\\
\neg S(c)\\
\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))\\
\hline
P(X) \lor Q(X)\\
\forall y.\ S(X)\\
\hline
\begin{array}{cc}
\begin{array}{c} P(a)\\ \forall y.\ S(a)\\ \hline \end{array} & \begin{array}{c} Q(a)\\ \forall y.\ S(a)\\ \hline \end{array}
\end{array}
\end{array}
\quad
\begin{array}{l}
\\ \\ \\ \gamma_\forall + \alpha_\land\\ \\ \\ \beta_\lor \\ \\ \\
\end{array}
$$

$$
\sigma = \{X \mapsto a\} \qquad \sigma = \{X \mapsto a\}
$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \vee Q(x)) \wedge \forall y.\ S(x))}{\begin{array}{c} P(X) \vee Q(X) \\ \forall y.\ S(X) \end{array}}\ \gamma_\forall + \alpha_\wedge$$

$$\beta_\vee$$

$$\begin{array}{c|c} \dfrac{\begin{array}{c} P(a) \\ \forall y.\ S(a) \end{array}}{\odot}\ \odot_\sigma & \dfrac{\begin{array}{c} Q(a) \\ \forall y.\ S(a) \end{array}}{} \end{array}$$

Closed

Solving Fairness Issues with Concurrency

$$
\begin{array}{c}
\neg P(a)\\
\neg Q(b)\\
\neg S(c)\\
\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\dfrac{\begin{array}{c}P(X) \lor Q(X)\\ \forall y.\ S(X)\end{array}}{\dfrac{\begin{array}{c}P(a)\\ \forall y.\ S(a)\end{array}}{\odot}\ \odot_\sigma \qquad \dfrac{\begin{array}{c}Q(a)\\ \boxed{\forall y.\ S(a)}\end{array}}{S(a)}\ \gamma_\forall}\ \beta_\lor}\ \gamma_\forall + \alpha_\land
\end{array}
$$

$$
\cfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))\end{array}}{\cfrac{\begin{array}{c}P(X) \lor Q(X)\\ \forall y.\ S(X)\end{array}}{\cfrac{\begin{array}{c}P(a)\\ \forall y.\ S(a)\end{array}}{\odot}\ \odot_\sigma \qquad \cfrac{\begin{array}{c}Q(a)\\ \forall y.\ S(a)\end{array}}{\cfrac{S(a)}{\cdots}}\ \gamma_\forall}\ \beta_\lor}\ \gamma_\forall + \alpha_\land
$$

Open

Solving Fairness Issues with Concurrency

$$
\cfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))\end{array}}{\begin{array}{c}P(X) \lor Q(X)\\ \forall y.\ S(X)\end{array}}\ \gamma_\forall + \alpha_\land
$$

$$
\cfrac{\begin{array}{c}P(X) \lor Q(X)\\ \forall y.\ S(X)\end{array}}{\begin{array}{c}P(b)\\ \forall y.\ S(b)\\ \hline\end{array} \qquad \begin{array}{c}Q(b)\\ \forall y.\ S(b)\\ \hline\end{array}}\ \beta_\lor
$$

$$
\sigma = \{X \mapsto b\} \qquad \sigma = \{X \mapsto b\}
$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\dfrac{\begin{array}{c} P(X) \lor Q(X) \\ \forall y.\ S(X) \end{array}}{\begin{array}{c} P(b) \\ \forall y.\ S(b) \\ \hline \end{array} \qquad \dfrac{\begin{array}{c} Q(b) \\ \forall y.\ S(b) \end{array}}{\odot}\ \odot_\sigma}\ \beta_\lor}\ \gamma_\forall + \alpha_\land$$

Closed

Solving Fairness Issues with Concurrency

$$
\begin{array}{c}
\neg P(a)\\
\neg Q(b)\\
\neg S(c)\\
\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\begin{array}{c} P(X) \lor Q(X) \\ \dfrac{\forall y.\ S(X)}{\begin{array}{cc} \begin{array}{c} P(b) \\ \dfrac{\forall y.\ S(b)}{S(b)}\ \gamma_\forall \end{array} & \begin{array}{c} Q(b) \\ \dfrac{\forall y.\ S(b)}{\odot}\ \odot_\sigma \end{array} \end{array}}\ \beta_\lor \end{array}}\ \gamma_\forall + \alpha_\land
\end{array}
$$

Solving Fairness Issues with Concurrency

$$
\begin{array}{c}
\neg P(a) \\
\neg Q(a) \\
\neg S(c) \\
\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\dfrac{\begin{array}{c} P(X) \lor Q(X) \\ \forall y.\ S(X) \end{array}}{\dfrac{\begin{array}{c} P(b) \\ \forall y.\ S(b) \end{array}}{\dfrac{S(b)}{\cdots}}\ \gamma_\forall \qquad \dfrac{\begin{array}{c} Q(b) \\ \forall y.\ S(b) \end{array}}{\odot}\ \odot_\sigma}\ \beta_\lor}\ \gamma_\forall + \alpha_\land
\end{array}
$$

Open

Solving Fairness Issues with Concurrency

$$
\begin{array}{cl}
\neg P(a) & \\
\neg Q(b) & \\
\neg S(c) & \\
\forall x.\, ((P(x) \lor Q(x)) \land \forall y.\, S(x)) & \\
\hline
P(X) \lor Q(X) & \gamma_{\forall} + \alpha_{\land} \\
\forall y.\, S(X) & \\
\hline
\begin{array}{c} P(X) \\ \forall y.\, S(X) \\ \hline \end{array} \qquad \begin{array}{c} Q(X) \\ \forall y.\, S(X) \\ \hline \end{array} & \beta_{\lor}
\end{array}
$$

$$
X \notin \{a, b\} \qquad\qquad X \notin \{a, b\}
$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(X) \lor Q(X)} \ \gamma_\forall + \alpha_\land$$

$$\dfrac{\forall y.\ S(X)}{P(X) \qquad\qquad Q(X)} \ \beta_\lor$$

$$\dfrac{\forall y.\ S(X)}{S(X)} \ \gamma_\forall \qquad\qquad \dfrac{\forall y.\ S(X)}{S(X)} \ \gamma_\forall$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\frac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{P(X) \lor Q(X)}\ \gamma_\forall + \alpha_\land$$

$$\frac{\forall y.\ S(X)}{\begin{array}{cc} P(X) & Q(X) \\ \dfrac{\forall y.\ S(X)}{\dfrac{S(X)}{\odot}\ \odot_\sigma}\ \gamma_\forall & \dfrac{\forall y.\ S(X)}{\dfrac{S(X)}{\odot}\ \odot_\sigma}\ \gamma_\forall \end{array}}\ \beta_\lor$$

$$\sigma = \{X \mapsto c\} \qquad \sigma = \{X \mapsto c\}$$

Solving Fairness Issues with Concurrency

$$
\begin{array}{c}
\neg P(a) \\
\neg Q(b) \\
\neg S(c) \\
\dfrac{\forall x.\ ((P(x) \lor Q(x)) \land \forall y.\ S(x))}{\begin{array}{c} P(c) \lor Q(c) \\ \dfrac{\forall y.\ S(c)}{P(c) \qquad\qquad\qquad Q(c)}\ \beta_{\lor} \end{array}}\ \gamma_{\forall} + \alpha_{\land}
\end{array}
$$

$$
\dfrac{\forall y.\ S(c)}{\dfrac{S(c)}{\odot}\ \odot_{\sigma}}\ \gamma_{\forall}
\qquad\qquad
\dfrac{\forall y.\ S(c)}{\dfrac{S(c)}{\odot}\ \odot_{\sigma}}\ \gamma_{\forall}
$$

$$
\sigma = \{X \mapsto c\} \qquad\qquad \sigma = \{X \mapsto c\}
$$

Contributions

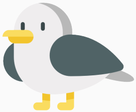
- A tableau-based proof-search procedure
- Concurrent exploration of branches
- Tackle fairness challenges
- Eager closure
- Backtrack and forbidden substitutions
- Completeness proof of the procedure
- Implemented into a tool: `Goéland`

3. The `Goéland` Automated Theorem Prover

The Goéland Tool

Proof-Search Procedure

- Concurrent proof-search procedure
- Eager closure
- Completeness proof



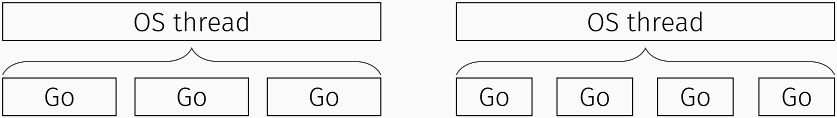
Additional Functionnalities

- Equality reasoning
- Deduction modulo theory
- Polymorphic types
- Alternative modes: incomplete, interactive, shared memory, ...
- Outputs: `Rocq`, `LambdaPi`, `Lisa` and `SC-TPTP`

The Goéland Tool

Implementation

- 40 000 lines of code
- Go programming language
- Designed for concurrency
- Goroutines: N:M lightweight threads



Theory Reasoning

Motivation and Challenges

- Reason within specific contexts (arithmetic, industrial problems, ...)
- Deal with a large number of axioms
- Handle multiple theories

Background Reasoners

- Equality
- Deduction modulo theory (DMT)

Deduction Modulo Theory

Principle

- Turns axioms into rewrite rules
- Triggers only relevant axioms
- Produces shorter proof
- Not limited to one theory

Main Heuristic

$(\forall \vec{x}.)\ A \Leftrightarrow F$ where:

- A is an atomic formula
- F is a non-atomic formula

Polarized DMT

$(\forall \vec{x}.)\ A \Rightarrow F$ where:

- A is an atomic formula
- F is a non-atomic formula

Axiom: $\forall x.\ P(x) \Leftrightarrow \forall y.\ Q(x, y) \land S(x, y)$

Rule: $P(X) \rightarrow \forall y.\ Q(X, y) \land S(X, y)$

Protocol of the Experiments

- Thousand of Problems for Theorem Provers (TPTP) library (v8.1.2)
- Syntactic (SYN) and set theory (SET) categories
- First-order logic (FOL)
- `Goéland` and its variants, `Zenon` (+ modulo), `Princess`, `Vampire` and `E`
- 300 seconds of timeout
- Intel Xeon E5-2680 v4 2.4GHz 2×14-core processor with 128GB

Goéland Variants over SYN and SET

	SYN (288 problems)		SET (464 problems)	
Goéland	**209**	(1.2 s)	124	(18.6 s)
Goéland+EQ	**213**	(0.3 s)	101	(15.6 s)
Goéland+DMT	**209**	(1.3 s)	**217**	(5.9 s)
Goéland+DMT+EQ	**213**	(0.5 s)	192	(10.2 s)
Goéland+DMT +Polarized	202	(0.3 s)	164	(1.5 s)

All Provers on FOF

	FOF (5396 problems)
`Goéland`	**613** (10 482 s — 17.1 s)
`Goéland+DMT`	**770** (6 935 s — 9 s)
`Goéland+DMT+EQ`	**801** (10 060 s — 12.5 s)
`Zenon`	**1 382** (9 026 s — 6.5 s)
`Zenon Modulo`	**1 389** (10 028 s — 7.2 s)
`Princess`	**1 621** (23 200 s — 14.3 s)
`Vampire`	**3 342** (42 873 s — 12.8 s)
`E`	**3 939** (39 638 s — 10.1 s)

Analysis

Promising Results and Features

- Deduction modulo theory
- Output proofs

Performances Issues

- Less problems solved than other ATP
- Memory management
- Equality reasoner
- Proof size

4. Production of Proof Certificate

4.1. Skolemization and Translation
4.2. A Deskolemization Strategy

Skolemization

Skolemization

- δ-rule
- $\exists$ and $\neg\forall$
- Introduces a *fresh* Skolem symbol
- The symbol is parametrized by the free variables of the branch

$$\exists z.\ P(z)$$

Skolemization

Skolemization

- δ-rule
- $\exists$ and $\neg\forall$
- Introduces a *fresh* Skolem symbol
- The symbol is parametrized by the free variables of the branch

$$\frac{\exists z.\ P(z)}{P(c)}\ \delta_{\exists}$$

Skolemization

Skolemization

- δ-rule
- $\exists$ and $\neg\forall$
- Introduces a *fresh* Skolem symbol
- The symbol is parametrized by the free variables of the branch

$$\begin{array}{c} \vdots \\ Q(X, Y) \\ \exists z.P(z) \end{array}$$

Skolemization

Skolemization

- δ-rule
- $\exists$ and $\neg\forall$
- Introduces a *fresh* Skolem symbol
- The symbol is parametrized by the free variables of the branch

$$\dfrac{\begin{array}{c}\vdots \\ Q(X, Y) \\ \exists z.P(z)\end{array}}{P(sko(X, Y))}\ \delta_{\exists}$$

Advanced Skolemization Strategies

Motivations

- Shorter proofs
- Faster proof search

Inner Skolemization (δ^+-rule)

- Extension of δ-rule
- Uses only the free variables of the formula
- Extensions: δ^{+^+}, δ^*, ...

$$\begin{array}{c} \vdots \\ Q(X, Y) \\ \exists z.\ P(z) \end{array}$$

Advanced Skolemization Strategies

Motivations

- Shorter proofs
- Faster proof search

Inner Skolemization (δ^+-rule)

- Extension of δ-rule
- Uses only the free variables of the formula
- Extensions: δ^{++}, δ^*, ...

$$\dfrac{\begin{array}{c} \vdots \\ Q(X, Y) \\ \exists z.\ P(z) \end{array}}{P(c)}\ \delta^+_\exists$$

Advanced Skolemization Strategies

Motivations

- Shorter proofs
- Faster proof search

Inner Skolemization (δ^+-rule)

- Extension of δ-rule
- Uses only the free variables of the formula
- Extensions: δ^{+^+}, δ^*, ...

$$\begin{gathered} \dots \\ Q(X, Y) \\ \exists z.\ P(z, X) \end{gathered}$$

Advanced Skolemization Strategies

Motivations

- Shorter proofs
- Faster proof search

Inner Skolemization (δ^+-rule)

- Extension of δ-rule
- Uses only the free variables of the formula
- Extensions: δ^{+^+}, δ^*, ...

$$\dfrac{\begin{array}{c}\vdots \\ Q(X, Y) \\ \exists z.\ P(z, X)\end{array}}{P(sko(X))}\ \delta^+_\exists$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\boxed{\neg D(f(X))}}\ \delta_{\neg\forall}}{}$$

$$\cfrac{\cfrac{\cfrac{\neg(D(X_2) \Rightarrow \forall y\ D(y))}{\boxed{D(X_2)}, \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\sigma = \{X_2 \mapsto f(X)\}}\ \odot_\sigma}{}\ \gamma_{\neg\exists}$$

(a) Outer Skolemization tableau.

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{\boxed{D(X)}, \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\boxed{\neg D(c)}}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_\sigma$$

(b) Inner Skolemization tableau.

Translation to Machine-Checkable Proofs

Gentzen-Schütte Calculus (GS3)

- Equivalent to tableaux: 1-to-1 mapping between rules
- Easily translatable to proof assistants
- No free variables

$$\begin{array}{cl}
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) & \\ \hline
\neg(D(X) \Rightarrow \forall y\ D(y)) & \gamma_{\neg\exists} \\ \hline
D(X), \neg(\forall y\ D(y)) & \alpha_{\neg\Rightarrow} \\ \hline
\neg D(f(X)) & \delta_{\neg\forall} \\ \hline
\neg(D(X_2) \Rightarrow \forall y\ D(y)) & \gamma_{\neg\exists} \\ \hline
D(X_2), \neg(\forall y\ D(y)) & \alpha_{\neg\Rightarrow} \\ \hline
\sigma = \{X_2 \mapsto f(X)\} & \odot_{\sigma}
\end{array}$$

(a) Outer Skolemization tableau proof.

$$\begin{array}{cl}
 & \\ \hline
\ldots, \neg D(c'), D(c'), \neg(\forall y\ D(y)) \vdash & \mathsf{ax} \\ \hline
\ldots, \neg(D(c') \Rightarrow \forall y\ D(y)) \vdash & \neg\Rightarrow \\ \hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \ldots, \neg D(c') \vdash & \neg\exists \\ \hline
\ldots, D(c), \neg(\forall y\ D(y)) \vdash & \neg\forall \\ \hline
\ldots, \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash & \neg\Rightarrow \\ \hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash & \neg\exists
\end{array}$$

(b) Equivalent GS3.

Outer Skolemization Only ☹

Why?

- δ-rule: the symbol has to be *fresh*
- γ-rule: must be instantiated by its final value
- Closure rule: unification with *any* term
- Problem: we use a term *before* its introduction

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_\sigma
$$

(a) Inner Skolemization tableau proof.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots, D(c), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \mathsf{ax}}{\ldots, D(c), \neg(\forall y\ D(y)) \vdash}\ \neg_\forall\ (\star)}{\ldots, \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_\exists
$$

(b) Incorrect equivalent GS3.

A Deskolemization Strategy

Idea

Perform all the Skolemization steps before the other rules, so the Skolem symbol is necessarily fresh.

Key Notion

- Formulas that *depend* on a Skolem symbol
- A formula F needs to be processed before another formula G iff G makes use of a Skolem symbol generated by F

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \boxed{\gamma_{\neg\exists}}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_\sigma$$

$$\frac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \boxed{\neg(D(c) \Rightarrow \forall y\ D(y))}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \boxed{\neg\exists}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y))}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \boxed{\alpha_{\neg\Rightarrow}}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \boxed{D(c)}, \boxed{\neg(\forall y\ D(y))} \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \boxed{\neg\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$
\begin{array}{c l}
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) & \\
\hline
\neg(D(X) \Rightarrow \forall y\ D(y)) & \gamma_{\neg\exists} \\
\hline
D(X), \neg(\forall y\ D(y)) & \alpha_{\neg\Rightarrow} \\
\hline
\neg D(c) & \delta^{+}_{\neg\forall} \\
\hline
\sigma = \{X \mapsto c\} & \odot_{\sigma}
\end{array}
$$

$$
\begin{array}{c l}
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash & \\
\hline\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash & \mathrm{W} \times 2 \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash & \neg\Rightarrow \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) & \neg\exists
\end{array}
$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists
$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$
\begin{array}{cl}
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) & \\
\hline
\neg(D(X) \Rightarrow \forall y\ D(y)) & \gamma_{\neg\exists} \\
\hline
D(X), \neg(\forall y\ D(y)) & \alpha_{\neg\Rightarrow} \\
\hline
\neg D(c) & \delta^{+}_{\neg\forall} \\
\hline
\sigma = \{X \mapsto c\} & \odot_{\sigma}
\end{array}
$$

$$
\begin{array}{cl}
 & \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash & \neg\exists \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash & \neg\forall \\
\hline\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash & \mathsf{W} \times 2 \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash & \neg\Rightarrow \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) & \neg\exists
\end{array}
$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \colorbox{pink}{$\neg(D(c) \Rightarrow \forall y\ D(y))$}, \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$
\begin{array}{c}
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \\
\hline
\neg(D(X) \Rightarrow \forall y\ D(y)) \quad \gamma_{\neg\exists} \\
\hline
D(X), \neg(\forall y\ D(y)) \quad \alpha_{\neg\Rightarrow} \\
\hline
\neg D(c) \quad \delta^{+}_{\neg\forall} \\
\hline
\sigma = \{X \mapsto c\} \quad \odot_{\sigma}
\end{array}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\neg\Rightarrow$}
\UnaryInfC{$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash$}
\RightLabel{$\neg\exists$}
\UnaryInfC{$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash$}
\RightLabel{$\neg\forall$}
\UnaryInfC{$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash$}
\doubleLine
\RightLabel{$\mathsf{W} \times 2$}
\UnaryInfC{$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash$}
\RightLabel{$\neg\Rightarrow$}
\UnaryInfC{$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash$}
\RightLabel{$\neg\exists$}
\UnaryInfC{$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))$}
\end{prooftree}
$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \mathsf{ax}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg_{\exists}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg_{\forall}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}}\ \neg_{\Rightarrow}
$$

Experiments

Implementation

- δ, δ^+ and δ^{+^+} Skolemization strategies
- `GS3` proofs
- Deskolemization algorithms

Evaluation Protocol

- Same setup as previous tests
- 3 Skolemization strategies + DMT
- Number of problems solved
- `Rocq` output
- Size of the proof (number of branches)

Results

	Problems Proved	Percentage Certified	Avg. Size Increase	Max. Size Increase
`Goéland`	261	100 %	0 %	-
`Goéland`+δ^+	272	100 %	8.1 %	5.3
`Goéland`+δ^{+^+}	274	100 %	10.6 %	10.3
`Goéland+DMT`	363	100 %	0 %	-
`Goéland+DMT`+δ^+	375	100 %	4.5 %	3.9
`Goéland+DMT`+δ^{+^+}	377	100 %	7.4 %	5.2

Contributions

- An optimization of the deskolemization algorithm for δ^+
- A deskolemization algorithm for δ^{++}
- Soundness proof for both translations
- Output of GS3 proof into `Rocq`, `LambdaPi`, `Lisa` and `SC-TPTP`
- Promising results
- 100% of the proofs are certified
- Far below the theoretical bound

Conclusion

Contributions

Goéland

- Fairness between branches managed by concurrency
- Completeness of the procedure
- Promising results (DMT)

Proof Certification

- A sound generic deskolemization algorithm
- Output of the proofs into `Rocq`, `LambdaPi`, `Lisa` and `SC-TPTP`

Future Work

`Goéland`

- Performance improvement (memory management, equality, Rust, ...)
- Heuristics, simulate "intuition" with learning methods, ...
- Modular and generic prover
- Non-classical logic & theories

Proof Certification

- Reduce the number of branches by the use of lemmas
- Integration of theories
- `SC-TPTP Utils` and proof elaboration
- Framework for verification of tableau proofs: `TableauxRocq`

Thank you! ☺

https://github.com/GoelandProver/Goeland

https://github.com/SC-TPTP/sc-tptp